# Math 275D Lecture 4 Notes

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## 1 Holder Continuity and Non-Lipschitz Continuity of Brownian Motion

### 1.1 Brownian Motion is Hölder Continuous on $\mathbb{Z}[\frac{1}{2}]$

Let's finish our proof of the following.

**Lemma 1.1.** *$B$ on $\mathbb{Z}[\frac{1}{2}]$ is uniformly continuous a.s.*

Last time, we had a sequence of events $H^N$ on which we had a condition bounding $|B(x) - B(y)|$.

*Proof.* In $H^N$, if $x, y \in \mathbb{Z}[\frac{1}{2}]$, and $|x - y| \leq 2^{-N}$, then $|B(x) - B(y)| \leq C2^{-\gamma N}$. We also know that $\mathbb{P}(H^N) \approx 1 - C2^{-\gamma N}$. Since $H^N = \bigcap_{m \geq N}^{\infty} G_m$, $H^N \subseteq H^{N+1}$. In $H^N$, for all $k \geq N$, if $|x - y| \in \mathbb{Z}[\frac{1}{2}]$ and $|x - y| \leq 2^{-k}$, then $|B(x) - B(y)| \leq 2^{-\gamma K}$. So in $H^N$, $B(t)$ is a uniformly continuous function.

Now, since $H^N \subseteq H^{N+1}$, we get $\mathbb{P}(\bigcup_{N=1}^{\infty} H^N) = \lim_{N \to \infty} \mathbb{P}(H^N) = 1$. So with probability 1, $B(t)$ is uniformly continuous on $\mathbb{Z}[\frac{1}{2}]$. $\square$

The proof actually shows the following:

**Corollary 1.1.** *For any $\gamma < \frac{1}{2}$, $B(t)$ on $\mathbb{Z}[\frac{1}{2}]$ is a $\gamma$-Hölder continuous function a.s.*

*Proof.* In $H^N$, if $|x - y| \leq 2^{-N}$, then $|B(x) - B(y)| \leq |x - y|^{\gamma} C$. So in $H^N$ for any $x, y \in [0, 1]$ with $x, y \in \mathbb{Z}[\frac{1}{2}]$, $|B(x) - B(y)| \leq |x - y|^{\gamma} C_N$. So in $\bigcup H_n$, $B(t)$ is $\gamma$-Hölder continuous.

How did we find $\gamma$? We had $\mathbb{E}[B(t)^4] \sim t^2$, so we chose $\gamma < 1/4$. We can do better by using the $2p$-th moment: $\mathbb{E}[B(t)^{2p}] \sim t^p$, so we can pick $\gamma < (p-1)/(2p)$. $\square$

Now that we have a uniformly continuous $B(t)$ on $\mathbb{Z}[\frac{1}{2}]$, we can extend it continuously to the entire positive real line.

## 1.2 Brownian motion is a.s. not Lipschitz anywhere

**Lemma 1.2.** *$B(t)$ is not Lipschitz at any point with probability 1.*

Here is the idea: Split $[0,1]$ into $n$ intervals of length $1/n$. When the distance between two points is $1/n$, then $B(x) - B(y)$ should be about $1/\sqrt{n}$

*Proof.* Fix a constant $C$. Let $A_n := \{\exists x \in [0,1]\text{s.t. if } |y-x| \leq 3/n, \text{ then } |\frac{B(y)-B(x)}{y-x}| \leq C\}$. We want to show that $\mathbb{P}(\bigcap_n A_n) = 0$, and we have that $A_{n+1} \subseteq A_n$. So we want to prove that $\lim_n \mathbb{P}(A_n) = 0$. Define $A_n^{(x)}$ to be the event $\{|y-x| \leq 3/n, |\frac{B(y)-B(x)}{y-x}| \leq C\}$ In $A_n^{(x)}$, $|B(x+1/n) - B(x)| \leq C/n$; call this event $A_{n,1}^{(x)}$. Similarly, $|B(x+2/n) - B(x+1/n)| \leq 3C/n$; call this event $A_{n,2}^{(x)}$. Going to the left of $x$, we also get $|B(x-1/n) - B(x)| \leq C/n$; call this event $A_{n,3}^{(x)}$. So we have $A_n^{(x)} \subseteq \bigcap_{i=1}^3 A_{n,i}^{(x)}$. Since the $A_{n,i}^{(x)}$ are independent,

$$\mathbb{P}(A_n) \leq \prod_{i=1}^{3} \mathbb{P}(A_{n,i}^{(x)})$$

We have that $\mathbb{P}(A_{n,i}^{(x)}) \sim 1/\sqrt{n}$, so $\mathbb{P}(A_n^{(x)}) \leq (1/n)^{3/2}$.

Let's extend this to all of $A_n$, not just a specific point. Let $a_{k,n} = k/n$. Then

$$\mathbb{P}\left(\bigcup_{k=1}^{n} A_n^{(a_{n,k})}\right) \leq n\mathbb{P}(A_n^{(a_{n,k})}) \sim \frac{1}{\sqrt{n}}.$$

The set of points where $x$ is Lipschitz is open, so if $B(t)$ is Lipschitz at $x$, then it is Lipschitz at $a_{n,k}$ for some $n, k$. So we can bound the probability of $A_n$; the details are left as an exercise.[1] □

## 1.3 $\sigma$-fields for Brownian motion

Which $\sigma$-field do we use for Brownian motion? We have a few choices we can use:

$$\mathcal{F}_s^0 = \sigma(B(t), t \leq s)$$
$$\mathcal{F}_s^+ = \bigcap_{t>s} \mathcal{F}_t^0$$

The second of these choices contains the first, and it contains events that allow you to "see into the future" a little bit. But we will see that the only extra events $\mathcal{F}_s^+$ contains are null sets. Here is an example of an event in $\mathcal{F}_s^+$ but not in $\mathcal{F}_s^0$:

$$A = \left\{\limsup_{t\to s^+} \frac{B(t) - B(s)}{t-s} \geq \frac{1}{2}\right\}.$$

[1] :(